Group-theoretical model of developed turbulence and renormalization of the Navier-Stokes equation

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On the basis of the Euler equation and its symmetry properties, this paper proposes a model of stationary homogeneous developed turbulence. A regularized averaging formula for the product of two fields is obtained. An equation for the averaged turbulent velocity field is derived from the Navier-Stokes equation by renormalization-group transformation.

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I. INTRODUCTION

There is a long history of the use of renormalization-group ideas in describing turbulence. In the profound books of McComb [1,2] and Frisch [3], one can find an extensive review on this subject. Among fundamental papers in the field are papers by Martin, Siggia, and Rose [4]; Forster, Nelson, and Stephen [5]; and Yakhot and Orszag [6]. The Navier-Stokes equation with random stirring forces is used as a mathematical model of turbulence in these fruitful but very sophisticated approaches. In contrast to these approaches, we propose here a mathematical model of developed turbulence that is based on special self-similar solutions of the Euler equation—therefore we do not need to introduce the external stirring forces. In our approach, we do not use the Fourier transform, perturbation series, or the sophisticated diagramic technique. The term "renormalization group" is broadly used in a variety of scenarios. At times, this term is used in situations where the transformation groups are not even considered. In our paper, we apply the idea of a renormalization group in its initial sense [7–9]. In relation to turbulence, we can *roughly* express Kadanoff's idea of "block picture" for the spin field in Ising's model as follows: If, instead of turbulent field $\boldsymbol{v}(\boldsymbol{r})$, we consider a field $\langle \boldsymbol{v}\rangle_{\sigma}(\boldsymbol{r})$ that is averaged on the scale σ, then the last one will "resemble" the original turbulent field $\boldsymbol{v}(\boldsymbol{r})$. The exact sense of "resemble" must be defined by the group of transformations for both fields and equations for these fields. Classical approaches [8,9] do not initially present the explicit expression for transformations from a renormalization group; they are provided, accurate to some unknown functions. Therefore, classical renormalization-group methods on the first step only exploit the fact that the symmetry exists. The transformations are then obtained in approximate form while considering the problem in the required range of the transformation parameter.

In this paper, solely on the basis of the Navier-Stokes equation, we propose a simple model of homogeneous stationary developed turbulence. This model allows us to explicitly derive the group of renormalization transformations. We average the Navier-Stokes equation over some small length scale σ_0 with the help of our averaging formula. Then we transform this averaged equation by a renormalization-group transformation to the equation for the average over any scale σ velocity field $\langle \boldsymbol{v}\rangle_{\sigma}(\boldsymbol{r},t)$, making use of symmetry properties of the base solution. Note that Frisch in his profound book [3] writes about applying the symmetries of Euler and Navier-Stokes equations in turbulence but he does not associate turbulence with the specific self-similar solutions of the Euler equation.

II. RENORMALIZED AVERAGING FORMULA

In different areas of physics (turbulence, kinetic theory, radiophysics, nonlinear dynamics), the averaged equations include the mean value of nonlinear terms. The fundamental problem in developing tractable quantitative theories is the closure of these equations. Specifically, the problem is obtaining the explicit expression for the averaged product of two functions in terms of their mean values. This section addresses this problem.

We derive the renormalized averaging formula on the basis of group properties of the specially chosen averaging procedure (Gaussian filtering). The method by which this formula has been obtained is similar to the method [10] used to rewrite (exactly) the Boltzmann collision integral in the divergence form. For simplicity, the analysis is initially performed for one space dimension and then extended for the general case of three (or more) dimensions.

Consider two functions $v(x)$ and $u(x)$ depending on one-dimensional real coordinate $x, -\infty < x < \infty$. The average of $v(x)$ is

$$\langle v\rangle(x) = \int_{-\infty}^{\infty} v(x')\Psi(x-x')dx', \qquad (1)$$

where $\Psi(x)$ is a weight function. When this function is Gaussian,

$$\Psi_{\sigma}(x) = \frac{1}{\sqrt{4\pi\sigma}}e^{-x^2/4\sigma}, \qquad (2)$$

the average is defined by Gauss transform (filtering),

$$\langle v\rangle(x) = \int_{-\infty}^{\infty} v(x')\Psi_{\sigma}(x-x')dx' = \hat{G}_{\sigma}v. \qquad (3)$$

The Gaussian distribution verifies the diffusion equation,

$$\frac{\partial}{\partial\sigma}\Psi_\sigma(x) = \nabla^2\Psi_\sigma(x), \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial x^2}, \tag{4}$$

thereby the Gauss transform operator (3) can be represented in the exponential form

$$\hat{G}_\sigma = e^{\sigma\nabla^2} \tag{5}$$

and has the following group properties:

$$\hat{G}_{\sigma_1}\hat{G}_{\sigma_2} = \hat{G}_{\sigma_1+\sigma_2}. \tag{6}$$

Consider the averaged product of two fields,

$$\langle vu\rangle(x) = \hat{G}_\sigma v(x)u(x) = e^{\sigma\nabla^2}v(x)u(x). \tag{7}$$

In this expression, the differentiation of the product of two multipliers can be done by the Leibnitz rule. To this end, decompose the differentiation operator ∇ into two parts,

$$\nabla = \nabla_1 + \nabla_2, \tag{8}$$

with differentiation operators ∇_1 and ∇_2 acting on the first $v(x)$ and the second $u(x)$ multipliers, respectively. Setting Eq. (8) in Eq. (7) gives

$$\begin{aligned} e^{\sigma\nabla^2}v(x)u(x) &= e^{\sigma(\nabla_1+\nabla_2)^2}v(x)u(x) \\ &= e^{2\sigma\nabla_1\nabla_2}e^{\sigma(\nabla_1^2+\nabla_2^2)}v(x)u(x) \\ &= e^{2\sigma\nabla_1\nabla_2}[e^{\sigma\nabla^2}v(x)][e^{\sigma\nabla^2}u(x)]. \end{aligned} \tag{9}$$

It then follows that, according to Eqs. (3) and (5), the average of the product of two fields can be written as

$$\langle vu\rangle = e^{2\sigma\nabla_1\nabla_2}\langle v\rangle\langle u\rangle. \tag{10}$$

Representing the operator of averaging of two fields products $e^{2\sigma\nabla_1\nabla_2}$ by its Taylor-series expansion yields the Leonard expression [11],

$$e^{2\sigma\nabla_1\nabla_2}\langle v\rangle\langle u\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}(2\sigma)^n\left(\frac{\partial^n}{\partial x^n}\langle v\rangle\right)\left(\frac{\partial^n}{\partial x^n}\langle u\rangle\right). \tag{11}$$

The formulas (10) and (11) allow us to express (at least formally) the mean of the product of the two functions through the mean values of multipliers. The simplest way of using Eq. (11) is by using only the first few terms, which provides a good approximation for averaging smooth functions. However, in the case of highly oscillating fields, such a simplification leads to substantial error, leading to the necessity of accounting, in principle, for all terms in Eq. (11).

Instead of using the Taylor-series expansion with an infinite number of terms (11) for the averaging operator $e^{2\sigma\nabla_1\nabla_2}$, we propose using the Taylor series with a residual term,

$$\begin{aligned} e^{2\sigma\nabla_1\nabla_2} = 1 &+ 2\sigma\nabla_1\nabla_2 + \frac{1}{2!}(2\sigma\nabla_1\nabla_2)^2 + \cdots \\ &+ \frac{1}{(n-1)!}(2\sigma\nabla_1\nabla_2)^{n-1} \\ &+ \frac{1}{n!}(2\sigma\nabla_1\nabla_2)^n\int_0^1 d\alpha q_n(\alpha)e^{2\alpha\sigma\nabla_1\nabla_2}, \end{aligned} \tag{12}$$

where n is an arbitrary natural number ($n=1, 2, 3,\ldots$) and

$$q_n(\alpha) = n(1-\alpha)^{n-1}, \quad \int_0^1 d\alpha q_n(\alpha) = 1. \tag{13}$$

Consider the case $n=1$. Then Eq. (12) is reduced to

$$e^{2\sigma\nabla_1\nabla_2} = 1 + 2\sigma\nabla_1\nabla_2\int_0^1 d\alpha e^{2\alpha\sigma\nabla_1\nabla_2}. \tag{14}$$

Taking into account that

$$e^{2\alpha\sigma\nabla_1\nabla_2+\sigma\nabla_1^2+\sigma\nabla_2^2} = e^{\alpha\sigma(\nabla_1+\nabla_2)^2+(1-\alpha)\sigma(\nabla_1^2+\nabla_2^2)} \tag{15}$$

yields

$$\langle vu\rangle_\sigma = \langle v\rangle_\sigma\langle u\rangle_\sigma + 2\sigma\int_0^1 d\alpha\langle\nabla\langle v\rangle_{(1-\alpha)\sigma}\nabla\langle u\rangle_{(1-\alpha)\sigma}\rangle_{\alpha\sigma}. \tag{16}$$

Substituting $\sigma'=(1-\alpha)\sigma$, expression (16) takes the following form:

$$\langle vu\rangle_\sigma = \langle v\rangle_\sigma\langle u\rangle_\sigma + 2\int_0^\sigma d\sigma'\langle\nabla\langle v\rangle_{\sigma'}\cdot\nabla\langle u\rangle_{\sigma'}\rangle_{\sigma-\sigma'}. \tag{17}$$

This averaging expression demonstrates that the integral part includes the contribution of all scales less than σ in an exact manner. Note that Eq. (17) is written for three dimensions by replacing $\nabla\nabla$ on the scalar product of three-dimensional nabla operators $\nabla\cdot\nabla$.

Two examples of applying the averaging formula.

First example

We decompose the velocity field into smooth and random components,

$$v_i(\boldsymbol{r}) = v_i^{sm} + v_i^{rnd}(\boldsymbol{r}), \tag{18}$$

where v_i^{sm} and $v_i^{rnd}(\boldsymbol{r})$ represent the smooth (on the scale less than σ, we consider v_i^{sm} to be constant, $\nabla_l v_i^{sm}=0$) and random parts, respectively. For the random part, we introduce the internal scale $\sigma_*\ll\sigma$, where the fluid viscosity damps out the velocity oscillations and $v_i^{rnd}(\boldsymbol{r})$ becomes a smooth function. Thus

$$\langle v_i^{rnd}\rangle_{\sigma'} = 0 \text{ if } \sigma' > \sigma_*, \tag{19}$$

$$\langle v_i^{rnd}\rangle_{\sigma'} = v_i^{rnd}(r) \text{ if } \sigma' < \sigma_*. \tag{20}$$

Assumptions (18)–(20) yield

$$\langle\nabla_l v_i\rangle_{\sigma'} = \nabla_l\langle v_i^{rnd}\rangle_{\sigma'} = 0 \text{ if } \sigma' > \sigma_*, \tag{21}$$

$$\langle\nabla_l v_i\rangle_{\sigma'} = \langle\nabla_l v_i^{rnd}\rangle_{\sigma'} = \nabla_l v_i \text{ if } \sigma' < \sigma_*. \tag{22}$$

The averaging formula (17) under the conditions (21) and (22) gives

$$\langle v_i v_k \rangle_\sigma = \langle v_i \rangle_\sigma \langle v_k \rangle_\sigma + 2\sigma_* \langle \nabla_l v_i \nabla_l v_k \rangle_\sigma, \tag{23}$$

where the repeated indices are summed. Hence, for the field (18)–(20), one obtains the expression for the internal scale σ_* through mean quantities (independently of averaging scale σ),

$$\sigma_* = \frac{\langle v^2 \rangle_\sigma - \langle v_i \rangle_\sigma \langle v_i \rangle_\sigma}{2\langle \nabla_l v_i \nabla_l v_i \rangle_\sigma}. \tag{24}$$

This expression is similar to the known turbulence expression for the Taylor microscale.

Second example

Consider the velocity field in the framework of the following model. Assume that in a region of a typical size L, the velocity field can be represented as

$$v_i(\boldsymbol{r}) = w_i + a_{ij} r_j + v_i^{rnd}(\boldsymbol{r}), \tag{25}$$

where w_i and a_{ij} are constants (they do not depend on r in the considered region but do change from one region to another) and $v_i^{rnd}(\boldsymbol{r})$ is an isotropic random field. This implies that the considered region is rectilinearly moving as a whole, rotating and homogeneously stretching. The liquid particles inside this region have random isotropic velocities. For the field (25), we have

$$\nabla_l \nabla_m v_i = \nabla_l \nabla_m v_i^{rnd}. \tag{26}$$

Using the averaging formula (10) and (12) with $n=2$ for the velocity field (25) yields

$$\langle v_i v_k \rangle_\sigma = \langle v_i \rangle_\sigma \langle v_k \rangle_\sigma + 2\sigma \nabla_l \langle v_i \rangle_\sigma \nabla_l \langle v_k \rangle_\sigma + 2\int_0^\sigma d\sigma' 2\sigma' \left\langle \left\langle \nabla_{l_1} \nabla_{l_2} v_i^{rnd} \right\rangle_{\sigma'} \left\langle \nabla_{l_1} \nabla_{l_2} v_k^{rnd} \right\rangle_{\sigma'} \right\rangle_{\sigma-\sigma'}, \tag{27}$$

where $l_1, l_2 = 1,2,3$. Accounting for isotropy of the random field, the integral term is proportional to the Kronecker delta δ_{ik},

$$\langle v_i v_k \rangle_\sigma = \langle v_i \rangle_\sigma \langle v_k \rangle_\sigma + 2\sigma \nabla_{l_1} \langle v_i \rangle_\sigma \nabla_{l_1} \langle v_k \rangle_\sigma + \frac{2}{3}\delta_{ik} \int_0^\sigma d\sigma' 2\sigma' \left\langle \left\langle \nabla_{l_1} \nabla_{l_2} v_{l_3}^{rnd} \right\rangle_{\sigma'} \left\langle \nabla_{l_1} \nabla_{l_2} v_{l_3}^{rnd} \right\rangle_{\sigma'} \right\rangle_{\sigma-\sigma'}. \tag{28}$$

Thus, for the model (25), the deviatoric part of the stress tensor can be expressed through the averaged velocity field in closed form,

$$\langle v_i v_k \rangle_\sigma - \frac{1}{3}\delta_{ik} \langle v^2 \rangle_\sigma = \langle v_i \rangle_\sigma \langle v_k \rangle_\sigma - \frac{1}{3}\delta_{ik} \langle v_{l_1} \rangle_\sigma \langle v_{l_1} \rangle_\sigma + 2\sigma \left(\nabla_{l_1} \langle v_i \rangle_\sigma \nabla_{l_1} \langle v_k \rangle_\sigma - \frac{1}{3}\delta_{ik} \nabla_{l_1} \langle v_{l_2} \rangle_\sigma \nabla_{l_1} \langle v_{l_2} \rangle_\sigma \right). \tag{29}$$

Let us make the following important remark regarding the averaging formula (10) and (12). Starting from Boussinesq, in papers devoted to deriving equations for averaged turbulent fields, it is usual to use the averaging formula on the basis of the gradient-diffusion hypothesis,

$$\langle v_i v_k \rangle_\sigma - \frac{1}{3}\delta_{ik} \langle v^2 \rangle_\sigma - \left\langle \langle v_i \rangle_\sigma \langle v_k \rangle_\sigma - \frac{1}{3}\delta_{ik} \langle \boldsymbol{v} \rangle_\sigma^2 \right\rangle_\sigma = -\nu_{tur} (\nabla_i \langle v_k \rangle_\sigma + \nabla_k \langle v_i \rangle_\sigma). \tag{30}$$

However, if the velocity field $v_i(\boldsymbol{r})$ in Eq. (30) is replaced by $\tilde{v}_i(\boldsymbol{r}) = -v_i(\boldsymbol{r})$, we obtain

$$\langle \tilde{v}_i \tilde{v}_k \rangle_\sigma - \frac{1}{3}\delta_{ik} \langle \tilde{v}^2 \rangle_\sigma - \left\langle \langle \tilde{v}_i \rangle_\sigma \langle \tilde{v}_k \rangle_\sigma - \frac{1}{3}\delta_{ik} \langle \tilde{\boldsymbol{v}} \rangle_\sigma^2 \right\rangle_\sigma = +\nu_{tur} (\nabla_i \langle \tilde{v}_k \rangle_\sigma + \nabla_k \langle \tilde{v}_i \rangle_\sigma). \tag{31}$$

Inevitably, from Eqs. (30) and (31), one of two fields $v_i(\boldsymbol{r},t)$ or $\tilde{v}_i(\boldsymbol{r},t)$ has negative turbulent viscosity. Thus, through averaging, it is impossible to derive the universal gradient-type formula of form (30). In other words, one cannot obtain the turbulent viscosity in equations for turbulence by simple averaging of the nonlinear term. The following sections show how the turbulent viscosity can be introduced in a natural way.

III. GROUP-THEORETICAL MODEL OF DEVELOPED TURBULENCE

Hereafter, we present a schematic description of the turbulent model, which is solely based on the Euler equation for incompressible flow,

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla \cdot \boldsymbol{v}\boldsymbol{v} - \nabla p, \quad \nabla \cdot \boldsymbol{v} = 0, \quad \rho = 1. \tag{32}$$

In some features, our further analysis will be similar to one used in [12]. The symmetry transformation properties of Eq. (32) are known. We will consider continuous symmetry transformations of Eq. (32) where the time variable is not transforming: scaling $\boldsymbol{r} \to e^{\tau\beta}\boldsymbol{r}, \boldsymbol{v} \to e^{\tau\beta}\boldsymbol{v}$; rotation $\boldsymbol{r} \to e^{\tau\boldsymbol{\Omega}\times}\boldsymbol{r}, \boldsymbol{v} \to e^{\tau\boldsymbol{\Omega}\times}\boldsymbol{v}$; and translation $\boldsymbol{r} \to \boldsymbol{r} + \tau\mathbf{v}$. These transformations constitute a group and induce the group of the velocity field transformations. The generators of its one-parameter subgroups are the operators that are the linear combination of scaling, translation, and rotation generators,

$$\hat{q}(\beta, \boldsymbol{\Omega}, \mathbf{v}) = \beta(1 - \boldsymbol{r} \cdot \nabla) - \mathbf{v} \cdot \nabla - \boldsymbol{\Omega} \cdot \boldsymbol{r} \times \nabla + (\boldsymbol{\Omega} \times). \tag{33}$$

Subgroup elements read as $\hat{g}_\tau = e^{\tau\hat{q}}$, where τ is a subgroup transformation parameter,

$$\hat{g}_{\tau_1} \hat{g}_{\tau_2} = \hat{g}_{\tau_1+\tau_2}. \tag{34}$$

The constants $\beta, \mathbf{v}, \boldsymbol{\Omega}$, as will be seen from Eqs. (37), have the following physical meaning: β is the rate (relative to parameter τ) of the homogeneous scaling, $\mathbf{v}$ is the velocity of translation, and $\boldsymbol{\Omega}$ is the angular velocity of rotation. The three generators $\hat{q}_1 = \hat{q}(\beta_1, \boldsymbol{\Omega}_1, \mathbf{v}_1)$, $\hat{q}_2 = \hat{q}(\beta_2, \boldsymbol{\Omega}_2, \mathbf{v}_2)$, and $\hat{q}_3 = \hat{q}(\beta_3, \boldsymbol{\Omega}_3, \mathbf{v}_3)$ with three different values of the constants

$\beta,\mathbf{\Omega},\mathbf{v}$ have the following commutation relations:

$$[\hat{q}_1,\hat{q}_2]=\hat{q}_3,\quad \beta_3=0,\quad \mathbf{\Omega}_3=\mathbf{\Omega}_1\times\mathbf{\Omega}_2,$$

$$\mathbf{v}_3=-\mathbf{v}_1\beta_2+\mathbf{v}_1\times\mathbf{\Omega}_2+(\beta_1+\mathbf{\Omega}_1\times)\mathbf{v}_2. \tag{35}$$

For further consideration, it is useful to present $e^{\tau\hat{q}}$ as a product of three transformations: scaling, rotation, and translation,

$$e^{\tau\hat{q}(\beta,\mathbf{\Omega},\mathbf{v})}=e^{\tau\beta(1-\boldsymbol{r}\cdot\nabla)}e^{\tau[(\mathbf{\Omega}\times)-\mathbf{\Omega}\cdot\boldsymbol{r}\times\nabla]}e^{-\boldsymbol{R}\cdot\nabla},$$

$$\boldsymbol{R}=\int_0^\tau d\tau e^{\tau(\beta+\mathbf{\Omega}\times)}\mathbf{v}. \tag{36}$$

An action of these operators on velocity field $\boldsymbol{v}(\boldsymbol{r})$ is as follows:

$$e^{\tau\beta(1-\boldsymbol{r}\cdot\nabla)}\boldsymbol{v}(\boldsymbol{r})=e^{\tau\beta}\boldsymbol{v}(e^{-\tau\beta}\boldsymbol{r}),$$

$$e^{\tau[(\mathbf{\Omega}\times)-\mathbf{\Omega}\cdot\boldsymbol{r}\times\nabla]}\boldsymbol{v}(\boldsymbol{r})=e^{\tau\mathbf{\Omega}\times}\boldsymbol{v}(e^{-\tau\mathbf{\Omega}\times}\boldsymbol{r}),$$

$$e^{-\boldsymbol{R}\cdot\nabla}\boldsymbol{v}(\boldsymbol{r})=\boldsymbol{v}(\boldsymbol{r}-\boldsymbol{R}). \tag{37}$$

The symmetry of Eqs. (32) is then expressed by

$$e^{\tau\hat{q}}\,\nabla\cdot\boldsymbol{v}\boldsymbol{v}=\nabla\cdot(e^{\tau\hat{q}}\boldsymbol{v})(e^{\tau\hat{q}}\boldsymbol{v}),\quad e^{\tau\hat{q}}\,\nabla p=\nabla p',$$

$$\nabla\cdot(e^{\tau\hat{q}}\boldsymbol{v})=0. \tag{38}$$

We propose to associate the phenomena of stationary turbulence with *self-similar solutions* of the Euler equation (32) in relevance to symmetry subgroups (33)–(35). The *self-similar solution* implies its dependence on time through the parameter of the space symmetry transformation only, i.e.,

$$\boldsymbol{v}(\boldsymbol{r},t)=e^{(t-t_0)\hat{q}}\boldsymbol{v}(\boldsymbol{r},t_0). \tag{39}$$

Application of the space symmetry transformation $e^{\tau\hat{q}}$ to this self-similar velocity field leads to its time translation,

$$e^{\tau\hat{q}}\boldsymbol{v}(\boldsymbol{r},t)=e^{(t+\tau-t_0)\hat{q}}\boldsymbol{v}(\boldsymbol{r},t_0)=\boldsymbol{v}(\boldsymbol{r},t+\tau). \tag{40}$$

Substituting Eq. (39) into the Euler equation (32) and using symmetry properties (38), we obtain an equation, which, at each time moment, defines the spatial configuration of the self-similar solution :

$$\hat{q}\boldsymbol{v}=-\nabla\cdot\boldsymbol{v}\boldsymbol{v}-\nabla p,\quad \nabla\cdot\boldsymbol{v}=0. \tag{41}$$

The evolution in time of a such self-similar solution $\boldsymbol{v}(\boldsymbol{r},t)$ is governed by the simple equation. Following Eq. (39), it reads

$$\frac{\partial\boldsymbol{v}}{\partial t}=\hat{q}\boldsymbol{v}. \tag{42}$$

Let us consider the space-time structure of self-similar velocity fields (39). Requiring solution (39) to be dynamically stationary, the velocity field $\boldsymbol{v}(\boldsymbol{r},t)$ must be represented by a superposition of time harmonics $e^{i\omega t}$ with real ω. Then it follows from Eq. (40) that expansion coefficients $\boldsymbol{v}_\omega(\boldsymbol{r})$ are eigenfields of the generator $\hat{q}$ corresponding to the pure imaginary eigenvalues

$$\boldsymbol{v}(\boldsymbol{r},t)=\boldsymbol{v}_{\omega=0}+\sum_\omega A_\omega(t)\boldsymbol{v}_\omega(\boldsymbol{r}), \tag{43}$$

$$A_\omega(t)=e^{i\omega(t-t_0)}A_\omega(t_0),\quad \hat{q}\boldsymbol{v}_\omega(\boldsymbol{r})=i\omega\boldsymbol{v}_\omega(\boldsymbol{r}),\quad -\infty<\omega<\infty. \tag{44}$$

Equation (44) indicates that in Eq. (43), only phases of coefficients A_ω change during the time evolution of turbulent velocity field $\boldsymbol{v}(\boldsymbol{r},t)$. Transformation (44) decorrelates phases of A_ω with different ω. The phases of A_ω are stochastic and, in principle, unknown for turbulent state. Therefore, only symmetry characteristics that are relevant to symmetry subgroup with generator (33) (rate of the homogeneous scaling, β; velocity of translation, $\mathbf{v}$; and angular velocity of rotation, $\mathbf{\Omega}$), the static component $\boldsymbol{v}_{\omega=0}$, and the power spectrum $|A_\omega|^2$ characterize the turbulent state in our model.

The following important statement emerges from this model: the turbulent field with parameters $\beta,\mathbf{\Omega},\mathbf{v}$ is invariant under transformation $e^{\tau\hat{q}(\beta,\mathbf{\Omega},\mathbf{v})}$ (accurate to the change of phases of coefficients A_ω, which are immaterial for the turbulent state),

$$e^{\tau\hat{q}(\beta,\mathbf{\Omega},\mathbf{v})}\boldsymbol{v}(\beta,\mathbf{\Omega},\mathbf{v})\overset{Av}{=}\boldsymbol{v}(\beta,\mathbf{\Omega},\mathbf{v}). \tag{45}$$

Indeed, applying the operator $e^{\tau\hat{q}(\beta,\mathbf{\Omega},\mathbf{v})}$ to the velocity field (43) leads only to the phase change of the decomposition coefficients $A_\omega(A_\omega\to e^{i\omega\tau}A_\omega)$. Hereafter, we will skip the symbol Av, and writing "invariance of the *turbulent velocity field*," the sentence "accurate to the change of stochastic phases" will be omitted for the sake of simplicity.

Let us now consider the averaged turbulent velocity fields. Taking into account that the commutator of ∇^2 with generators $\hat{q}$ is

$$[\nabla^2,\hat{q}]=-2\beta\nabla^2, \tag{46}$$

the Lie algebra of generators $\hat{q}(\beta,\mathbf{\Omega},\mathbf{v})$ is extended by adding the generator ∇^2 to it. As a result, we obtain the algebra $\{\tau\hat{q}(\beta,\mathbf{\Omega},\mathbf{v}),\sigma\nabla^2\}$ and the corresponding transformation group $e^{\tau\hat{q}(\beta,\mathbf{\Omega},\mathbf{v})+\sigma\nabla^2}$, which we will call the Euler equation renormalization group. From Eq. (46), one derives

$$e^{\tau\hat{q}}e^{\sigma\nabla^2}e^{-\tau\hat{q}}=e^{(e^{2\tau\beta}\sigma)\nabla^2}, \tag{47}$$

$$e^{\sigma\nabla^2}e^{\tau\hat{q}}e^{-\sigma\nabla^2}=e^{\tau(\hat{q}-2\beta\sigma\nabla^2)}. \tag{48}$$

The operator $e^{\tau(\hat{q}-2\beta\sigma\nabla^2)}$ can be represented as a product of two operators in two forms,

$$e^{\tau(\hat{q}-2\beta\sigma\nabla^2)}=e^{\sigma(1-e^{2\tau\beta})\nabla^2}e^{\tau\hat{q}}=e^{\tau\hat{q}}e^{\sigma(e^{-2\tau\beta}-1)\nabla^2}. \tag{49}$$

We introduce an averaged velocity field using the Gauss weight function with scale σ,

$$\langle\boldsymbol{v}\rangle_\sigma=e^{\sigma\nabla^2}\boldsymbol{v}(\boldsymbol{r})=\frac{1}{(4\pi\sigma)^{3/2}}\int d\boldsymbol{r}'\,e^{-(\boldsymbol{r}-\boldsymbol{r}')^2/4\sigma}\boldsymbol{v}(\boldsymbol{r}'). \tag{50}$$

The invariance (45) for turbulent fields provides the corresponding invariance for averaged turbulent fields $\langle\boldsymbol{v}\rangle_\sigma$. Using Eqs. (48), (50), and (45) we obtain the expression of the renormalization-group symmetry transformation for averaged fields,

$$e^{\tau(\hat{q}-2\beta\sigma\nabla^2)}\langle \boldsymbol{v}\rangle_\sigma = e^{\sigma\nabla^2}e^{\tau\hat{q}}e^{-\sigma\nabla^2}\langle \boldsymbol{v}\rangle_\sigma = e^{\sigma\nabla^2}e^{\tau\hat{q}}\boldsymbol{v} = e^{\sigma\nabla^2}\boldsymbol{v} = \langle \boldsymbol{v}\rangle_\sigma. \tag{51}$$

Using Eq. (49), this equation can be rewritten in an equivalent form,

$$e^{\tau_\sigma\hat{q}}\langle \boldsymbol{v}\rangle_{\sigma_0} = \langle \boldsymbol{v}\rangle_\sigma, \tag{52}$$

where

$$\tau_\sigma = \frac{1}{\beta}\ln\sqrt{\frac{\sigma}{\sigma_0}} \tag{53}$$

and

$$\tau_\sigma\hat{q} = \ln\sqrt{\frac{\sigma}{\sigma_0}}(1-\boldsymbol{r}\cdot\nabla) + \frac{1}{\beta}\ln\sqrt{\frac{\sigma}{\sigma_0}} \times[-\mathbf{v}\cdot\nabla - \boldsymbol{\Omega}\cdot\boldsymbol{r}\times\nabla + (\boldsymbol{\Omega}\times)]. \tag{54}$$

Therefore, for stationary homogeneous turbulence with *symmetry characteristics* $\beta,\boldsymbol{\Omega},\mathbf{v}$, the change of the scale of averaging from σ_0 to σ is equivalent to the composition of scaling, rotation, and translation transformations. Note also that the scaling coefficient $\sqrt{\sigma/\sigma_0}$ in Eq. (54) depends only on initial σ_0 and final σ scales of averaging and does not depend on turbulence parameters $\beta,\boldsymbol{\Omega},\mathbf{v}$. We call the property (51) and (52) a renormalization-group invariance of averaged turbulent fields.

Strictly speaking, the velocity fields (43) describe real turbulent flows only *locally*. In different regions of the flow, the symmetry characteristics $\beta,\boldsymbol{\Omega},\mathbf{v}$ may be different. It is remarkable that (as will be seen later), the renormalization procedure, developed in Sec. IV, does not depend on specific values of $\beta,\boldsymbol{\Omega},\mathbf{v}$ [only the *fact* of a renormalization-group invariance (52) is used]. If $\beta,\boldsymbol{\Omega},\mathbf{v}$ are slowly varying in the physical space (and in time), this procedure can be applied to an inhomogeneous case also.

IV. RENORMALIZATION OF THE AVERAGED NAVIER-STOKES EQUATION

We will study the case when characteristics of turbulence are changing in space and time slowly. Consider now the Navier-Stokes equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla\cdot\boldsymbol{v}\boldsymbol{v} + \nabla p - \nu\nabla^2\boldsymbol{v} = 0, \quad \nabla\cdot\boldsymbol{v} = 0, \quad \rho = 1. \tag{55}$$

First, we average this equation using a Gauss weight function with some small scale σ_0. Concerning this scale, we assume that σ_0 is the inner threshold of the turbulence such that, on scales larger than $\sigma_0, \sigma > \sigma_0$, we have Euler turbulence with self-similarity according to Eq. (52). On the scales of order or less than $\sigma_0, \sigma \leqslant \sigma_0$, we assume that the fluid viscosity destroys the internal symmetry structure of turbulence (52) and the velocity pulsations become purely isotropic (25). Therefore, acting on Eq. (55) by the Gauss transform $e^{\sigma_0\nabla^2}$ and making use of the averaging formula in form (29), we have

$$\frac{\partial\langle \boldsymbol{v}\rangle_{\sigma_0}}{\partial t} + \nabla\cdot(\langle \boldsymbol{v}\rangle_{\sigma_0}\langle \boldsymbol{v}\rangle_{\sigma_0} + 2\sigma_0\nabla_l\langle \boldsymbol{v}\rangle_{\sigma_0}\nabla_l\langle \boldsymbol{v}\rangle_{\sigma_0}) + \nabla p' = \nu\nabla^2\langle \boldsymbol{v}\rangle_{\sigma_0},$$

$$\nabla\cdot\langle \boldsymbol{v}\rangle_{\sigma_0} = 0. \tag{56}$$

The exact determination of the inner threshold scale σ_0 of turbulence is an open problem. Here we propose a simple method of estimating its value. The gradients of the velocity field tend to destroy the continuous structure of flow due to nonlinear steeping while the viscosity processes smooth over the flow. Equate these two factors on the basis of dimensional consideration to derive the equation for σ_0. The antisymmetric part, which is connected with the rotation as a whole, must be excluded from gradient $\nabla\boldsymbol{v}$,

$$\frac{\nu}{\sigma_0} = c[\langle S_{ik}\rangle_{\sigma_0}\langle S_{ik}\rangle_{\sigma_0}]^{1/2}, \tag{57}$$

where

$$\langle S_{ik}\rangle_\sigma = \frac{1}{2}(\nabla_i\langle v_k\rangle_\sigma + \nabla_k\langle v_i\rangle_\sigma) \tag{58}$$

and c is some unknown constant. From Eq. (57), we obtain the expression for the inner threshold scale of turbulence,

$$\sigma_0 = \frac{\nu}{c[\langle S_{ik}\rangle_{\sigma_0}\langle S_{ik}\rangle_{\sigma_0}]^{1/2}}. \tag{59}$$

Excluding molecular viscosity ν from Eq. (56) with the help of Eq. (57), we have

$$\frac{\partial\langle \boldsymbol{v}\rangle_{\sigma_0}}{\partial t} + \nabla\cdot(\langle \boldsymbol{v}\rangle_{\sigma_0}\langle \boldsymbol{v}\rangle_{\sigma_0} + 2\sigma_0\nabla_l\langle \boldsymbol{v}\rangle_{\sigma_0}\nabla_l\langle \boldsymbol{v}\rangle_{\sigma_0}) + \nabla p' = \nu_{tur}(\sigma_0)\nabla^2\langle \boldsymbol{v}\rangle_{\sigma_0}, \tag{60}$$

where

$$\nu_{tur}(\sigma_0) = c\sigma_0[\langle S_{ik}\rangle_{\sigma_0}\langle S_{ik}\rangle_{\sigma_0}]^{1/2} = \nu. \tag{61}$$

Equation (60) has two additional terms in comparison with the Euler equation (32). Under the transformation $e^{\tau\hat{q}}$, they transform as follows:

$$e^{\tau\hat{q}}\nabla\cdot(\nabla_l\langle \boldsymbol{v}\rangle_{\sigma_0}\nabla_l\langle \boldsymbol{v}\rangle_{\sigma_0}) = e^{2\tau\beta}\nabla\cdot\left[\nabla_l(e^{\tau\hat{q}}\langle \boldsymbol{v}\rangle_{\sigma_0})\nabla_l(e^{\tau\hat{q}}\langle \boldsymbol{v}\rangle_{\sigma_0})\right], \tag{62}$$

$$e^{\tau\hat{q}}\nu_{tur}(\sigma_0)e^{-\tau\hat{q}} = e^{-2\tau\beta}\nu_{tur}(e^{2\tau\beta}\sigma_0), \tag{63}$$

$$e^{\tau\hat{q}}\nabla^2\langle \boldsymbol{v}\rangle_{\sigma_0} = e^{2\tau\beta}\nabla^2(e^{\tau\hat{q}}\langle \boldsymbol{v}\rangle_{\sigma_0}). \tag{64}$$

With this result, when transformation $e^{\tau_\sigma\hat{q}}$ with $\tau_\sigma = (1/\beta)\ln\sqrt{\sigma/\sigma_0}$ is applied to Eq. (56), along with the invariance property (52), we obtain the final equation for the averaged turbulent field for any scale $\sigma > \sigma_0$,

$$\frac{\partial\langle \boldsymbol{v}\rangle_\sigma}{\partial t} + \boldsymbol{\nabla}\cdot(\langle \boldsymbol{v}\rangle_\sigma\langle \boldsymbol{v}\rangle_\sigma + 2\sigma\nabla_l\langle \boldsymbol{v}\rangle_\sigma\nabla_l\langle \boldsymbol{v}\rangle_\sigma) + \boldsymbol{\nabla} p' = \nu_{tur}(\sigma)\boldsymbol{\nabla}^2\langle \boldsymbol{v}\rangle_\sigma,$$

$$\nu_{tur}(\sigma) = c\sigma[\langle S_{ik}\rangle_\sigma\langle S_{ik}\rangle_\sigma]^{1/2}, \quad \boldsymbol{\nabla}\cdot\langle \boldsymbol{v}\rangle_\sigma = 0. \tag{65}$$

The equations for averaged fields (65) are similar to the known equations of Leonard [13] applied in the LES approach [14] (the small difference is that the dissipative term with molecular viscosity is, in principle, not present here). It is important to stress that we have shown that the turbulent viscosity appeared not as a result of averaging of the nonlinear term in the Navier-Stokes equation, but from the molecular viscosity term with the help of renormalization-group transformation.

The renormalization-group symmetry of our model provided invariance of Eq. (65) under the scale transformation, which involves changing the averaging scale σ,

$$e^{\ln\,\alpha[r\cdot\boldsymbol{\nabla}+2\sigma(\partial/\partial\sigma)-1]}\langle \boldsymbol{v}\rangle_\sigma(\boldsymbol{r},t) = \frac{1}{\alpha}\langle \boldsymbol{v}\rangle_{\alpha^2\sigma}(\alpha\boldsymbol{r},t), \tag{66}$$

where α is a scaling parameter.

V. CONCLUDING REMARKS

In this work, we obtained the regularized averaging formula for averaging a two velocity field product. Assuming that on the small length scale σ_0 (inner threshold of the turbulence), the turbulent velocity field can be approximated as the sum of a smooth velocity field and a random *isotropic* field, we averaged the Navier-Stokes equation over this small scale by making use of our averaging formula. Solely on the basis of the Euler equation and its symmetry properties, we proposed the model of stationary homogeneous turbulence. We proposed to associate the phenomena of stationary turbulence with the special self-similar solutions (43) of the Euler equation (32)—they represent the linear superposition of eigenfields of the symmetry subgroup generators corresponding to the pure imaginary eigenvalues. From this model, in particular, it follows that for stationary homogeneous turbulence, the change of the scale of averaging from σ_0 to σ is equivalent to the composition of scaling, rotation, and translation transformations. We call this property a renormalization-group invariance of averaged turbulent fields. Because the Navier-Stokes equation is invariant under translations and rotations, the renormalization group invariance provides an opportunity to transform the averaged Navier-Stokes equation over a small scale σ_0 to any scale σ by simple scaling.

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